

Tensor power sequences and the approximation of tensor product operators

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December 23, 2016

Abstract

The approximation numbers of the L_2 -embedding of mixed order Sobolev functions on the d -torus are well studied. They are given as the nonincreasing rearrangement of the d -th tensor power of the approximation number sequence in the univariate case. I present results on the asymptotic and preasymptotic behavior for tensor powers of arbitrary sequences of polynomial decay. This can be used to study the approximation numbers of many other tensor product operators, like the embedding of mixed order Sobolev functions on the d -cube into $L_2([0, 1]^d)$ or the embedding of mixed order Jacobi functions on the d -cube into $L_2([0, 1]^d, w_d)$ with Jacobi weight w_d .

1 Introduction and Results

Let $(\sigma_n)_{n \in \mathbb{N}}$ be a nonincreasing zero sequence. Its d -th tensor power is the sequence $(\sigma_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}^d}$, where $\sigma_{\mathbf{n}} = \prod_{j=1}^d \sigma_{n_j}$ for $\mathbf{n} \in \mathbb{N}^d$ and $d \in \mathbb{N}$. Since there are only finitely many indices $\mathbf{n} \in \mathbb{N}^d$ with $\sigma_{\mathbf{n}} \geq \varepsilon$ for any $\varepsilon > 0$, we can rearrange it to a nonincreasing zero sequence $(a_{n,d})_{n \in \mathbb{N}}$.

Tensor power sequences like this occur naturally in the study of approximation numbers of tensor power operators. If $(\sigma_n)_{n \in \mathbb{N}}$ is the sequence of approximation numbers of a compact operator between two Hilbert spaces, then $(a_{n,d})_{n \in \mathbb{N}}$ is the sequence of approximation numbers of the compact d -th tensor power operator between the tensor power spaces. For instance, the embeddings of $H_{\text{mix}}^{s_1}(G^d)$ into $H_{\text{mix}}^{s_2}(G^d)$ for $s_1 > s_2$ are of this type. Here, G is a compact manifold and $H_{\text{mix}}^s(G^d)$ is the Hilbert space of real or complex-valued functions on G^d with dominating mixed smoothness $s \in \mathbb{R}$ and integrability 2.

Motivated by this example, we will assume that $(\sigma_n)_{n \in \mathbb{N}}$ is of polynomial decay. What can we say about the behavior of $(a_{n,d})_{n \in \mathbb{N}}$? In Section 2 we will prove in particular the following asymptotic result.

Theorem 1. *Let $(\sigma_n)_{n \in \mathbb{N}}$ be a nonincreasing zero sequence, $s > 0$ and for $d \in \mathbb{N}$ let $(a_{n,d})_{n \in \mathbb{N}}$ be the nonincreasing rearrangement of its d -th tensor power.*

(i) *If $\sigma_n = \mathcal{O}(n^{-s})$, then $a_{n,d} = \mathcal{O}(n^{-s} (\log n)^{s(d-1)})$.*

(ii) *If $\sigma_n = \Omega(n^{-s})$, then $a_{n,d} = \Omega(n^{-s} (\log n)^{s(d-1)})$.*

(iii) *If the limit $c = \lim_{n \rightarrow \infty} \sigma_n n^s$ exists, then the limit $c_d = \lim_{n \rightarrow \infty} \frac{a_{n,d} n^s}{(\log n)^{s(d-1)}}$ also exists and $c_d = \frac{c^d}{(d-1)!^s}$.*

This theorem is inspired by [KSU15], where Kühn, Sickel and Ullrich prove the existence of the limit of $\left(a_{n,d} \cdot n^s (\log n)^{-s(d-1)}\right)_{n \in \mathbb{N}}$ in an interesting special case and compute it. There, $(a_{n,d})_{n \in \mathbb{N}}$ is the sequence of approximation numbers for the L_2 -embedding of the tensor power space $H_{\text{mix}}^s(\mathbb{T}^d)$ on the d -torus $[0, 2\pi]^d$, equipped with different natural norms. It gives us a pretty good understanding of the asymptotic behavior of the d -th tensor power $(a_{n,d})_{n \in \mathbb{N}}$ of a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of polynomial decay. If σ_n is roughly $c n^{-s}$ for large n , then $a_{n,d}$ is roughly $\frac{c^d}{(d-1)!^s} n^{-s} (\log n)^{s(d-1)}$ for n larger than a certain threshold.

But even for modest values of d , the size of this threshold may go far beyond the scope of computational capabilities. Indeed, while $a_{n,d}$ decreases, the function $n^{-s} (\log n)^{s(d-1)}$ grows rapidly as n goes from 1 to e^{d-1} . For $n^{-s} (\log n)^{s(d-1)}$ to become less than 1, n even has to be super exponentially large in d . Thus, any estimate for the sequence $(a_{n,d})_{n \in \mathbb{N}}$ in terms of $n^{-s} (\log n)^{s(d-1)}$ is useless to describe its behavior in the range $n \leq 2^d$, the so called preasymptotic behavior of the tensor power sequence. In fact, the preasymptotic behavior of the tensor power sequence $(a_{n,d})_{n \in \mathbb{N}}$ for large powers (or dimensions) d is very different from its asymptotic behavior. In Section 3, we will prove the following preasymptotic estimate.

Theorem 2. *Let $(\sigma_n)_{n \in \mathbb{N}}$ be a nonincreasing zero sequence and let $(a_{n,d})_{n \in \mathbb{N}}$ be the nonincreasing rearrangement of its d -th tensor power. Let $\sigma_1 = 1$, $\sigma_2 > 0$ and $v = \#\{n \geq 2 \mid \sigma_n = \sigma_2\}$ and assume that $(\sigma_n)_{n \in \mathbb{N}}$ satisfies $\sigma_n \leq C n^{-s}$ for some $s, C > 0$ and all $n \geq 2$. For any $d \in \mathbb{N}$ and $n \in \mathbb{N}$ with $vd < n \leq (1+v)^d$,*

$$\sigma_2 \cdot \left(\frac{1}{n}\right)^{\frac{\log \sigma_2^{-1}}{\log\left(1 + \frac{d}{\log 2}\right)}} \leq a_{n,d} \leq \left(\frac{\exp(C^{2/s})}{n}\right)^{\frac{\log \sigma_2^{-1}}{\log(\sigma_2^{-2/s} d)}}.$$

This shows that the tensor power sequence, which roughly decays like n^{-s} for large values of n , only decays roughly like n^{-t_d} with $t_d = \frac{\log \sigma_2^{-1}}{\log d}$ for small n , if d is large. This is why I will refer to t_d as the preasymptotic rate of the tensor power sequence. It generalizes to $t_d = \frac{\log(\sigma_1/\sigma_2)}{\log d}$ for arbitrary values of σ_1 . Apparently, the preasymptotic rate is much worse than the asymptotic rate.

An interesting consequence of these preasymptotic estimates is the following tractability result. For each $d \in \mathbb{N}$, let $T^{(d)}$ be a compact norm-one operator between two Hilbert spaces and let T_d be its d -th tensor power. Assume that the corresponding approximation numbers $a_n(T^{(d)})$ are nonincreasing in d and that $a_n(T^{(1)})$ decays polynomially in n . Then the problem of approximating T_d by linear functionals is strongly polynomially tractable, iff it is polynomially tractable, iff $a_2(T^{(d)})$ decays polynomially in d .

In Section 4, these results will be applied to the L_2 -approximation of mixed order Sobolev functions on the d -torus, as well as mixed order Jacobi and Sobolev functions on the d -cube, taking different normalizations into account. For instance, we will consider the d -variate Sobolev space $H_{\text{mix}}^s([0, 1]^d)$ of dominating mixed smoothness $s \in \mathbb{N}$, equipped with the scalar product

$$\langle f, g \rangle = \sum_{\alpha \in \{0, \dots, s\}^d} \langle D^\alpha f, D^\alpha g \rangle_{L_2} \quad (1.1)$$

and the subspace $H_{\text{mix}}^s(\mathbb{T}^d)$ of periodic functions. Let us denote the corresponding sequences of L_2 -approximation numbers by $(a_{n,d})_{n \in \mathbb{N}}$ and $(\tilde{a}_{n,d})_{n \in \mathbb{N}}$.

An application of Theorem 1 yields the asymptotic constants

$$\lim_{n \rightarrow \infty} \frac{a_{n,d} \cdot n^s}{(\log n)^{s(d-1)}} = \lim_{n \rightarrow \infty} \frac{\tilde{a}_{n,d} \cdot n^s}{(\log n)^{s(d-1)}} = (\pi^d \cdot (d-1)!)^{-s}. \quad (1.2)$$

In particular, $\tilde{a}_{n,d}$ and $a_{n,d}$ do not only have the same rate of convergence, but even their ratio tends to one as n tends to infinity. This means that the L_2 -approximation of mixed order Sobolev functions on the d -cube with n linear functionals is just as hard for nonperiodic functions as for periodic functions, if n is large enough.

The preasymptotic rate \tilde{t}_d for the periodic case satisfies

$$\frac{s \cdot \log(2\pi)}{\log d} \leq \tilde{t}_d \leq \frac{s \cdot \log(2\pi) + 1}{\log d}. \quad (1.3)$$

Although this is significantly worse than the asymptotic main rate s , it still grows linearly with the smoothness. An increasing dimension can hence be neutralized by increasing the smoothness of the functions. In contrast, the preasymptotic rate t_d for the nonperiodic case satisfies

$$\frac{1.2803}{\log d} \leq t_d \leq \frac{1.2825}{\log d} \quad (1.4)$$

for any $s \geq 2$. This means that increasing the smoothness of the functions beyond $s = 2$ in the nonperiodic setting is a very ineffective way of reducing the approximation error. The L_2 -approximation of mixed order Sobolev functions on the d -cube with less than 2^d linear functionals is hence much harder for nonperiodic functions than for periodic functions.

This is also reflected in the corresponding tractability results: The approximation problem $(H_{\text{mix}}^{s_d}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d))_{d \in \mathbb{N}}$ is (strongly) polynomially tractable, iff the smoothness s_d grows at least logarithmically with the dimension, whereas the approximation problem $(H_{\text{mix}}^{s_d}([0, 1]^d) \hookrightarrow L_2([0, 1]^d))_{d \in \mathbb{N}}$ is never (strongly) polynomially tractable. A similar effect for functions with coordinatewise increasing smoothness has already been observed by Papageorgiou and Woźniakowski in [PW10]. However, the tractability result for the space of periodic functions heavily depends on the side length $b - a$ of the torus $\mathbb{T}^d = [a, b]^d$. If it is less than 2π , (strong) polynomial tractability is equivalent to logarithmic increase of the smoothness. If it equals 2π , (strong) polynomial tractability is equivalent to polynomial increase of the smoothness. If it is larger than 2π , there cannot be (strong) polynomial tractability. These tractability results and interpretations can be found in Section 5.

2 Asymptotic Behavior of Tensor Power Sequences

Let $(\sigma_n)_{n \in \mathbb{N}}$ be a nonincreasing zero sequence of polynomial decay. We want to analyze the asymptotic behavior of the nonincreasing rearrangement $(a_{n,d})_{n \in \mathbb{N}}$ of its d -th tensor power. If $(\sigma_n)_{n \in \mathbb{N}}$ is the sequence of L_2 -approximation numbers of $H^s(\mathbb{T})$, a sequence of order n^{-s} , this is the sequence of L_2 -approximation numbers of $H_{\text{mix}}^s(\mathbb{T}^d)$. It is known that there are positive constants C and c such that

$$a_{n,d} \leq C \cdot n^{-s} (\log n)^{s(d-1)} \quad \text{and} \quad (2.1)$$

$$a_{n,d} \geq c \cdot n^{-s} (\log n)^{s(d-1)} \quad (2.2)$$

for any $n \geq 2$ in this case. This result goes back to Babenko [B60] and Mityagin [M62] and the early 1960s. In [SU09], Sickel and Ullrich prove the same statement for the sequence $(a_{n,d})_{n \in \mathbb{N}}$ of approximation numbers of the embedding of $H_{\text{mix}}^s([0,1]^d)$ into $L_2([0,1]^d)$. In fact, the upper bound holds for the rearranged d -th tensor power of any sequence in $\mathcal{O}(n^{-s})$, whereas the lower bound holds for the rearranged d -th tensor power of any sequence in $\Omega(n^{-s})$.

But one can be more precise. Fix some $s > 0$ and let us consider the quantities

$$\begin{aligned} C_1 &= \limsup_{n \rightarrow \infty} \sigma_n n^s, & c_1 &= \liminf_{n \rightarrow \infty} \sigma_n n^s, \\ C_d &= \limsup_{n \rightarrow \infty} \frac{a_{n,d} \cdot n^s}{(\log n)^{s(d-1)}}, & c_d &= \liminf_{n \rightarrow \infty} \frac{a_{n,d} \cdot n^s}{(\log n)^{s(d-1)}}. \end{aligned}$$

The limits C_d and c_d can be interpreted as asymptotic or optimal constants for the bounds (2.1) and (2.2). For any $C > C_d$ respectively $c < c_d$ there is a threshold $n_0 \in \mathbb{N}$ such that (2.1) respectively (2.2) holds for all $n \geq n_0$, whereas for any $C < C_d$ respectively $c > c_d$ there is no such threshold. In this section, we will prove that

$$\frac{c_1^d}{(d-1)!^s} \leq c_d \leq C_d \leq \frac{C_1^d}{(d-1)!^s}, \quad (2.3)$$

where equality can but does not always hold. It is remarkable that for any input sequence of polynomial decay the asymptotic constants of its d -th tensor power sequences decay super exponentially in d . Note that the given proof also provides a possibility to track down admissible thresholds n_0 for any $C > \frac{C_1^d}{(d-1)!^s}$ respectively any $c < \frac{c_1^d}{(d-1)!^s}$.

In case the sequence $(\sigma_n n^s)_{n \in \mathbb{N}}$ has a limit $c > 0$, relation (2.3) implies that the sequence $\left(a_{n,d} n^s (\log n)^{-s(d-1)}\right)_{n \in \mathbb{N}}$ also has a limit and this limit is $\frac{c^d}{(d-1)!^s}$. In other words, if σ_n is roughly $c n^{-s}$ for large n , then $a_{n,d}$ is roughly $\frac{c^d}{(d-1)!^s} n^{-s} (\log n)^{s(d-1)}$ for large n .

In order to study the asymptotic behavior of the tensor power sequence, it will be essential to study the asymptotic behavior of the cardinalities

$$A_N(r, l) = \# \left\{ \mathbf{n} \in \{N, N+1, \dots\}^l \mid \prod_{j=1}^l n_j \leq r \right\} \quad (2.4)$$

for $l \in \{1, \dots, d\}$ and $N \in \mathbb{N}$ as $r \rightarrow \infty$. In [KSU15, Lemma 3.2], it is shown that

$$r \left(\frac{(\log \frac{r}{2^l})^{l-1}}{(l-1)!} - \frac{(\log \frac{r}{2^l})^{l-2}}{(l-2)!} \right) \leq A_2(r, l) \leq r \frac{(\log r)^{l-1}}{(l-1)!} \quad (2.5)$$

for $l \geq 2$ and $r \in \{4^l, 4^l + 1, \dots\}$. Consequently, we have

$$\lim_{r \rightarrow \infty} \frac{A_N(r, l)}{r (\log r)^{l-1}} = \frac{1}{(l-1)!} \quad (2.6)$$

for $N = 2$. In fact, (2.6) holds true for any $N \in \mathbb{N}$. This can be derived from the case $N = 2$, but for the reader's convenience, I will give a complete proof.

Lemma 1.

$$\lim_{r \rightarrow \infty} \frac{A_N(r, l)}{r (\log r)^{l-1}} = \frac{1}{(l-1)!}.$$

Proof. Note that for all values of the parameters,

$$A_N(r, l+1) = \sum_{k=N}^{\infty} A_N\left(\frac{r}{k}, l\right), \quad (2.7)$$

where $A_N\left(\frac{r}{k}, l\right) = 0$ for $k > \frac{r}{N^l}$. This allows a proof by induction on $l \in \mathbb{N}$.

Like in estimate (2.5), we first show that

$$A_2(r, l) \leq r \frac{(\log r)^{l-1}}{(l-1)!} \quad (2.8)$$

for any $l \in \mathbb{N}$ and $r \geq 1$. This is obviously true for $l = 1$. On the other hand, if this relation holds for some $l \in \mathbb{N}$ and if $r \geq 1$, then

$$\begin{aligned} A_2(r, l+1) &= \sum_{k=2}^{\lfloor r \rfloor} A_2\left(\frac{r}{k}, l\right) \leq \sum_{k=2}^{\lfloor r \rfloor} \frac{r (\log \frac{r}{k})^{l-1}}{k (l-1)!} \\ &\leq \frac{r}{(l-1)!} \int_1^r \frac{(\log \frac{r}{x})^{l-1}}{x} dx = \frac{r}{(l-1)!} \left[-\frac{1}{l} \left(\log \frac{r}{x} \right)^l \right]_1^r = r \frac{(\log r)^l}{l!} \end{aligned} \quad (2.9)$$

and (2.8) is proven. In particular, we have

$$\limsup_{r \rightarrow \infty} \frac{A_N(r, l)}{r (\log r)^{l-1}} \leq \frac{1}{(l-1)!} \quad (2.10)$$

for $l \in \mathbb{N}$ and $N = 2$. Clearly, the same holds for $N \geq 2$, since $A_N(r, l)$ is decreasing

in N . Relation (2.10) for $N = 1$ follows from the case $N = 2$ with the identity

$$\begin{aligned} A_1(r, l) &= \sum_{m=0}^l \# \left\{ \mathbf{n} \in \mathbb{N}^l \mid \# \{1 \leq j \leq l \mid n_j \neq 1\} = m \wedge \prod_{j=1}^d n_j \leq r \right\} \\ &= \mathbf{1}_{r \geq 1} + \sum_{m=1}^l \binom{l}{m} \cdot A_2(r, m). \end{aligned} \quad (2.11)$$

It remains to prove that

$$\liminf_{r \rightarrow \infty} \frac{A_N(r, l)}{r (\log r)^{l-1}} \geq \frac{1}{(l-1)!} \quad (2.12)$$

for $N \in \mathbb{N}$ and $l \in \mathbb{N}$. Again, this is obvious for $l = 1$. Suppose, (2.12) holds for some $l \in \mathbb{N}$ and let $b < 1$. Then there is some $r_0 \geq 1$ such that

$$A_N(r, l) \geq br \frac{(\log r)^{l-1}}{(l-1)!} \quad (2.13)$$

for all $r \geq r_0$ and hence

$$\begin{aligned} A_N(r, l+1) &\geq \sum_{k=N}^{\lfloor r/r_0 \rfloor} A_N\left(\frac{r}{k}, l\right) \geq \sum_{k=N}^{\lfloor r/r_0 \rfloor} \frac{br (\log \frac{r}{k})^{l-1}}{k (l-1)!} \\ &\geq \frac{br}{(l-1)!} \int_N^{\frac{r}{r_0}} \frac{(\log \frac{r}{x})^{l-1}}{x} dx = \frac{br}{l!} \left(\left(\log \frac{r}{N} \right)^l - (\log r_0)^l \right) \geq b^2 r \frac{(\log r)^l}{l!} \end{aligned} \quad (2.14)$$

for sufficiently large r . Since this is true for any $b < 1$, the induction step is complete. \square

Theorem 3. *Let $(\sigma_n)_{n \in \mathbb{N}}$ be a nonincreasing zero sequence. For any $d \in \mathbb{N}$ and $s > 0$ the nonincreasing rearrangement $(a_{n,d})_{n \in \mathbb{N}}$ of its d -th tensor power satisfies:*

$$\begin{aligned} (i) \quad & \limsup_{n \rightarrow \infty} \frac{a_{n,d} \cdot n^s}{(\log n)^{s(d-1)}} \leq \frac{\left(\limsup_{n \rightarrow \infty} \sigma_n n^s \right)^d}{(d-1)!^s}, \\ (ii) \quad & \liminf_{n \rightarrow \infty} \frac{a_{n,d} \cdot n^s}{(\log n)^{s(d-1)}} \geq \frac{\left(\liminf_{n \rightarrow \infty} \sigma_n n^s \right)^d}{(d-1)!^s}. \end{aligned}$$

Proof. Without loss of generality, we can assume that $s = 1$ and $\sigma_1 = 1$. If $\sigma_1 \neq 0$ and $s \neq 1$ the stated inequalities follow from the corresponding inequalities for the sequence $(\tilde{\sigma}_n)_{n \in \mathbb{N}}$ with $\tilde{\sigma}_n = (\sigma_n/\sigma_1)^{1/s}$. If $\sigma_1 = 0$, they are trivial.

Proof of (i): Assume without loss of generality that $C := \limsup_{n \rightarrow \infty} \sigma_n n$ is finite and let $C_3 > C_2 > C_1 > C$. Then there is some $N \in \mathbb{N}$ such that

$$\sigma_n \leq \frac{C_1}{n} \quad (2.15)$$

for any $n \geq N$. We want to prove that

$$C_d := \limsup_{n \rightarrow \infty} \left(a_{n,d} \frac{n}{(\log n)^{d-1}} \right) \leq \frac{C^d}{(d-1)!} \quad (2.16)$$

for any $d \in \mathbb{N}$. Since $n/(\log n)^{d-1}$ is finally increasing, instead of giving an upper bound for $a_{n,d}$ in terms of n , we can just as well give an upper bound for n in terms of $a_{n,d}$ to obtain (2.16). Clearly, there are at least n elements in the tensor power sequence greater than or equal to $a_{n,d}$ and hence

$$\begin{aligned} n &\leq \# \{ \mathbf{n} \in \mathbb{N}^d \mid \sigma_{\mathbf{n}} \geq a_{n,d} \} \\ &= \sum_{l=0}^d \# \{ \mathbf{n} \in \mathbb{N}^d \mid \# \{ 1 \leq j \leq d \mid \mathbf{n}_j \geq N \} = l \wedge \sigma_{\mathbf{n}} \geq a_{n,d} \} \\ &\leq \sum_{l=0}^d \binom{d}{l} N^{d-l} \# \{ \mathbf{n} \in \{N, N+1, \dots\}^l \mid \sigma_{\mathbf{n}} \geq a_{n,d} \}. \end{aligned} \quad (2.17)$$

For every \mathbf{n} in the last set, relation (2.15) implies that $\prod_{j=1}^d \mathbf{n}_j \leq \frac{C_1^l}{a_{n,d}}$. Thus,

$$n \leq \sum_{l=0}^d \binom{d}{l} N^{d-l} A_N \left(\frac{C_1^l}{a_{n,d}}, l \right). \quad (2.18)$$

Lemma 1 yields that, if n and hence $\frac{C_1^l}{a_{n,d}}$ is large enough,

$$A_N \left(\frac{C_1^l}{a_{n,d}}, l \right) \leq \frac{C_2^l}{a_{n,d}} \frac{\left(\log \frac{C_2^l}{a_{n,d}} \right)^{l-1}}{(l-1)!} \quad (2.19)$$

for $l \in \{1, \dots, d\}$. Letting $n \rightarrow \infty$, the term for $l = d$ is dominant and hence

$$n \leq \frac{C_3^d}{a_{n,d}} \frac{\left(\log \frac{C_3^d}{a_{n,d}} \right)^{d-1}}{(d-1)!} \quad (2.20)$$

for large values of n . By the monotonicity of $n/(\log n)^{d-1}$, we obtain

$$a_{n,d} \frac{n}{(\log n)^{d-1}} \leq \frac{C_3^d}{(d-1)!} \cdot \left(\frac{\log(C_3^d a_{n,d}^{-1})}{\log\left(\frac{C_3^d}{(d-1)!} (\log(C_3^d a_{n,d}^{-1}))^{d-1} a_{n,d}^{-1}\right)} \right)^{d-1}. \quad (2.21)$$

The fraction in brackets tends to one as n and hence $a_{n,d}^{-1}$ tends to infinity and thus

$$C_d = \limsup_{n \rightarrow \infty} \left(a_{n,d} \frac{n}{(\log n)^{d-1}} \right) \leq \frac{C_3^d}{(d-1)!}. \quad (2.22)$$

Since this is true for any $C_3 > C$, the proof of (2.16) is complete.

Proof of (ii): Assume without loss of generality that $c := \liminf_{n \rightarrow \infty} \sigma_n n$ is positive and let $0 < c_3 < c_2 < c_1 < c$. Then there is some $N \in \mathbb{N}$ such that

$$\sigma_n \geq \frac{c_1}{n} \quad (2.23)$$

for any $n \geq N$. We want to prove that

$$c_d := \liminf_{n \rightarrow \infty} \left(a_{n,d} \frac{n}{(\log n)^{d-1}} \right) \leq \frac{c^d}{(d-1)!^s} \quad (2.24)$$

for any $d \in \mathbb{N}$. Clearly, there are at most $n-1$ elements in the tensor power sequence greater than $a_{n,d}$ and hence

$$n > \# \{ \mathbf{n} \in \mathbb{N}^d \mid \sigma_{\mathbf{n}} > a_{n,d} \} \geq \# \{ \mathbf{n} \in \{N, N+1, \dots\}^d \mid \sigma_{\mathbf{n}} > a_{n,d} \}. \quad (2.25)$$

Relation (2.23) implies that every $\mathbf{n} \in \{N, N+1, \dots\}^d$ with $\prod_{j=1}^d n_j < \frac{c_1^d}{a_{n,d}}$ is contained in the last set. This observation and Lemma 1 yield that

$$n > A_N \left(\frac{c_2^d}{a_{n,d}}, d \right) \geq \frac{c_3^d}{a_{n,d}} \frac{\left(\log \frac{c_3^d}{a_{n,d}} \right)^{d-1}}{(d-1)!} \quad (2.26)$$

for sufficiently large $n \in \mathbb{N}$. By the monotonicity of $n/(\log n)^{d-1}$ for large n , we obtain

$$a_{n,d} \frac{n}{(\log n)^{d-1}} \geq \frac{c_3^d}{(d-1)!} \cdot \left(\frac{\log(c_3^d a_{n,d}^{-1})}{\log\left(\frac{c_3^d}{(d-1)!} (\log(c_3^d a_{n,d}^{-1}))^{d-1} a_{n,d}^{-1}\right)} \right)^{d-1}. \quad (2.27)$$

The fraction in brackets tends to one as n and hence $a_{n,d}^{-1}$ tends to infinity and thus

$$c_d = \liminf_{n \rightarrow \infty} \left(a_{n,d} \frac{n}{(\log n)^{d-1}} \right) \geq \frac{c_3^d}{(d-1)!}. \quad (2.28)$$

Since this is true for any $c_3 < c$, the proof of (2.24) is complete. \square

Note that Theorem 1, as stated in the introduction, is an immediate consequence of Theorem 3. Part (iii) of Theorem 1 also says that there must be equality in both relations of Theorem 3, if the limit of $\sigma_n n^s$ for $n \rightarrow \infty$ exists. In that sense, the bounds of Theorem 3 are sharp. It is natural to ask, whether equality always holds true. The answer is no, as shown by the following example.

Example 1. The sequence $(\sigma_n)_{n \in \mathbb{N}}$, defined by $\sigma_n = 2^{-k}$ for $n \in \{2^k, \dots, 2^{k+1} - 1\}$ and $k \in \mathbb{N}_0$, decays linearly in n , but is constant on segments of length 2^k . It satisfies

$$\begin{aligned} C &:= \limsup_{n \rightarrow \infty} \sigma_n n = \lim_{k \rightarrow \infty} 2^{-k} \cdot (2^{k+1} - 1) = 2, \\ c &:= \liminf_{n \rightarrow \infty} \sigma_n n = \lim_{k \rightarrow \infty} 2^{-k} \cdot 2^k = 1. \end{aligned} \quad (2.29)$$

Also the values of $(a_{n,d})_{n \in \mathbb{N}}$ for $d \in \mathbb{N}$ are of the form 2^{-k} for some $k \in \mathbb{N}_0$, where

$$\begin{aligned} \# \{n \in \mathbb{N} \mid a_{n,d} = 2^{-k}\} &= \sum_{\mathbf{k} \in \mathbb{N}_0^d: |\mathbf{k}|=k} \# \{\mathbf{n} \in \mathbb{N}^d \mid \sigma_{n_j} = 2^{-k_j} \text{ for } j = 1 \dots d\} \\ &= \sum_{\mathbf{k} \in \mathbb{N}_0^d: |\mathbf{k}|=k} 2^k = 2^k \cdot \binom{k+d-1}{d-1} = \frac{2^k}{(d-1)!} \cdot (k+1) \cdot \dots \cdot (k+d-1). \end{aligned} \quad (2.30)$$

Hence, $a_{n,d} = 2^{-k}$ for $N(k-1, d) < n \leq N(k, d)$ with $N(-1, d) = 0$ and

$$N(k, d) = \sum_{j=0}^k \frac{2^j}{(d-1)!} \cdot (j+1) \cdot \dots \cdot (j+d-1) \quad (2.31)$$

for $k \in \mathbb{N}_0$. The monotonicity of $n/(\log n)^{d-1}$ for large n implies

$$\begin{aligned} C_d &:= \limsup_{n \rightarrow \infty} \frac{a_{n,d} \cdot n}{(\log n)^{d-1}} = \lim_{k \rightarrow \infty} \frac{2^{-k} \cdot N(k, d)}{(\log N(k, d))^{d-1}} \quad \text{and} \\ c_d &:= \liminf_{n \rightarrow \infty} \frac{a_{n,d} \cdot n}{(\log n)^{d-1}} = \lim_{k \rightarrow \infty} \frac{2^{-k} \cdot N(k-1, d)}{(\log N(k-1, d))^{d-1}}. \end{aligned} \quad (2.32)$$

We insert the relations

$$\begin{aligned} N(k, d) &\leq \frac{(k+d)^{d-1}}{(d-1)!} \sum_{j=0}^k 2^j \leq \frac{2^{k+1} \cdot (k+d)^{d-1}}{(d-1)!} \quad \text{and} \\ N(k, d) &\geq \frac{(k-l)^{d-1}}{(d-1)!} \sum_{j=k-l+1}^k 2^j = \frac{2^{k+1} \cdot (k-l)^{d-1}}{(d-1)!} \cdot (1 - 2^{-l}) \end{aligned} \quad (2.33)$$

for arbitrary $l \in \mathbb{N}$ in (2.32) and obtain

$$C_d = 2 \cdot \frac{(\log_2 e)^{d-1}}{(d-1)!} \quad \text{and} \quad c_d = \frac{(\log_2 e)^{d-1}}{(d-1)!}. \quad (2.34)$$

In particular, for $d \neq 1$ we have a chain of strict inequalities:

$$\frac{c^d}{(d-1)!} < c_d < C_d < \frac{C^d}{(d-1)!}. \quad (2.35)$$

More generally, the tensor product of d nonincreasing zero sequences $(\sigma_{n,j})_{n \in \mathbb{N}}$ is the sequence $(\sigma_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}^d}$, where $\sigma_{\mathbf{n}} = \prod_{j=1}^d \sigma_{n_j, j}$ for $\mathbf{n} \in \mathbb{N}^d$ and $d \in \mathbb{N}$. It can be rearranged to a nonincreasing zero sequence $(a_{n,d})_{n \in \mathbb{N}}$. An example of such a sequence is given by the L_2 -approximation numbers of Sobolev functions on the d -torus with mixed order $(s_1, \dots, s_d) \in \mathbb{R}_+^d$. They are generated by the L_2 -approximation numbers of the univariate Sobolev spaces $H^{s_j}(\mathbb{T})$, which are of order n^{-s_j} . It is known that $(a_{n,d})_{n \in \mathbb{N}}$ has the order $n^{-s} (\log n)^{s(l-1)}$ in this case, where s is the minimum among all numbers s_j and l is its multiplicity. This was proven by Mityagin [M62] for integer vectors (s_1, \dots, s_d) and by Nikol'skaya [N74] in the general case. See [T86, pp. 32, 36, 72] for more details. It is not hard to deduce that the rearranged tensor product sequence is in $\Omega \left(n^{-s} (\log n)^{s(l-1)} \right)$ respectively $\mathcal{O} \left(n^{-s} (\log n)^{s(l-1)} \right)$ for any choice of factor sequences $(\sigma_{n,j})_{n \in \mathbb{N}}$ in $\Omega(n^{-s_j})$ respectively $\mathcal{O}(n^{-s_j})$. But in contrast to the tensor power case, asymptotic constants of tensor product sequences in general are not determined by the asymptotic constants of the factor sequences.

Example 2. Consider the sequences $(\sigma_n)_{n \in \mathbb{N}}$, $(\mu_n)_{n \in \mathbb{N}}$ and $(\tilde{\mu}_n)_{n \in \mathbb{N}}$ with

$$\sigma_n = n^{-1}, \quad \mu_n = n^{-2}, \quad \tilde{\mu}_n = \begin{cases} 1, & \text{for } n \leq N, \\ n^{-2}, & \text{for } n > N, \end{cases} \quad (2.36)$$

for $n \in \mathbb{N}$ and some $N \in \mathbb{N}$. The tensor product $(\sigma_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}^2}$ of $(\sigma_n)_{n \in \mathbb{N}}$ and $(\mu_n)_{n \in \mathbb{N}}$

has the form

$$\sigma_{\mathbf{n}} = \mathbf{n}_1^{-1} \mathbf{n}_2^{-2} \quad \text{for } \mathbf{n} \in \mathbb{N}^2 \quad (2.37)$$

and its nonincreasing rearrangement $(a_n)_{n \in \mathbb{N}}$ satisfies for all $n \in \mathbb{N}$ that

$$\begin{aligned} n &\leq \# \{ \mathbf{n} \in \mathbb{N}^2 \mid \sigma_{\mathbf{n}} \geq a_n \} = \# \{ \mathbf{n} \in \mathbb{N}^2 \mid \mathbf{n}_1 \mathbf{n}_2^2 \leq a_n^{-1} \} \\ &\leq \sum_{\mathbf{n}_2=1}^{\infty} \# \{ \mathbf{n}_1 \in \mathbb{N} \mid \mathbf{n}_1 \leq a_n^{-1} \mathbf{n}_2^{-2} \} \leq a_n^{-1} \sum_{\mathbf{n}_2=1}^{\infty} \mathbf{n}_2^{-2} \leq 2a_n^{-1}, \end{aligned} \quad (2.38)$$

and hence

$$\limsup_{n \rightarrow \infty} a_n n \leq 2. \quad (2.39)$$

The tensor product $(\tilde{\sigma}_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}^2}$ of $(\sigma_n)_{n \in \mathbb{N}}$ and $(\tilde{\mu}_n)_{n \in \mathbb{N}}$ takes the form

$$\tilde{\sigma}_{\mathbf{n}} = \begin{cases} \mathbf{n}_1^{-1}, & \text{if } \mathbf{n}_2 \leq N, \\ \mathbf{n}_1^{-1} \mathbf{n}_2^{-2}, & \text{else,} \end{cases} \quad (2.40)$$

and its nonincreasing rearrangement $(\tilde{a}_n)_{n \in \mathbb{N}}$ satisfies for all $n \in \mathbb{N}$ that

$$n \geq \# \{ \mathbf{n} \in \mathbb{N}^2 \mid \tilde{\sigma}_{\mathbf{n}} > \tilde{a}_n \} \geq N \# \{ \mathbf{n}_1 \in \mathbb{N} \mid \mathbf{n}_1^{-1} > \tilde{a}_n \} \geq N (\tilde{a}_n^{-1} - 1) \quad (2.41)$$

and thus

$$\liminf_{n \rightarrow \infty} \tilde{a}_n n \geq N. \quad (2.42)$$

Hence, matching asymptotic constants of the factor sequences do not necessarily lead to matching asymptotic constants of the tensor product sequences.

3 Preasymptotic Behavior of Tensor Power Sequences

In order to estimate the size of $a_{n,d}$ for small values of n , we give explicit estimates of $A_2(r, l)$ for $l \leq d$ and small values of r . The right asymptotic behavior of these estimates, however, is less important. Note that $A_2(r, l) = 0$ for $r < 2^l$.

Lemma 2. *Let $r \geq 0$ and $l \in \mathbb{N}$. For any $\delta > 0$ we have*

$$\begin{aligned} A_2(r, l) &\leq \frac{r^{1+\delta}}{\delta^{l-1}} && \text{and} \\ A_2(r, l) &\geq \frac{r}{3 \cdot 2^{l-1}} && \text{for } r \geq 2^l. \end{aligned}$$

Proof. Both estimates hold in the case $l = 1$, since

$$A_2(r, 1) = \begin{cases} 0, & \text{for } r < 2, \\ \lfloor r \rfloor - 1, & \text{for } r \geq 2. \end{cases} \quad (3.1)$$

If they hold for some $l \in \mathbb{N}$, then

$$\begin{aligned} A_2(r, l+1) &= \sum_{k=2}^{\infty} A_2\left(\frac{r}{k}, l\right) \leq \frac{r^{1+\delta}}{\delta^{l-1}} \sum_{k=2}^{\infty} \frac{1}{k^{1+\delta}} \\ &\leq \frac{r^{1+\delta}}{\delta^{l-1}} \int_1^{\infty} \frac{1}{x^{1+\delta}} dx = \frac{r^{1+\delta}}{\delta^l} \end{aligned} \quad (3.2)$$

and for $r \geq 2^{l+1}$

$$A_2(r, l+1) \geq A_2\left(\frac{r}{2}, l\right) \geq \frac{r/2}{3 \cdot 2^{l-1}} = \frac{r}{3 \cdot 2^l}. \quad (3.3)$$

We have thus proven Lemma 2 by induction. \square

Theorem 4. Let $(\sigma_n)_{n \in \mathbb{N}}$ be a nonincreasing zero sequence with $\sigma_1 = 1$ and $\sigma_2 > 0$ and let $(a_{n,d})_{n \in \mathbb{N}}$ be the nonincreasing rearrangement of its d -th tensor power.

- (i) If $(\sigma_n)_{n \in \mathbb{N}}$ satisfies $\sigma_n \leq C n^{-s}$ for some $s, C > 0$ and all $n \geq 2$, then $(a_{n,d})_{n \in \mathbb{N}}$ satisfies for any $d, n \in \mathbb{N}$ and $\delta \in (0, 1]$

$$\begin{aligned} a_{n,d} &\leq \left(\frac{\tilde{C}(\delta)}{n} \right)^{\alpha(d,\delta)} \quad \text{with} \\ \tilde{C}(\delta) &= \exp\left(\frac{C^{(1+\delta)/s}}{\delta} \right) \quad \text{and} \quad \alpha(d,\delta) = \frac{\log \sigma_2^{-1}}{\log \left(\sigma_2^{-(1+\delta)/s} \cdot d \right)}. \end{aligned}$$

- (ii) Let $v = \#\{n \geq 2 \mid \sigma_n = \sigma_2\}$. Then $(a_{n,d})_{n \in \mathbb{N}}$ satisfies for any $d \in \mathbb{N}$ and $n \in \mathbb{N}$ with $vd < n \leq (1+v)^d$

$$a_{n,d} \geq \sigma_2 \cdot \left(\frac{1}{n} \right)^{\beta(d,n)} \quad \text{with} \quad \beta(d,n) = \frac{\log \sigma_2^{-1}}{\log \left(1 + \frac{v}{\log_{1+v} n} \cdot d \right)}.$$

Again, Theorem 2 is an immediate consequence of Theorem 4. Note that $a_{n,d} = \sigma_2$ for $1 < n \leq vd$. The additional assumption that $\sigma_1 = 1$ was made to reduce the complexity of the estimates. One can easily obtain bounds for arbitrary values of σ_1 by applying Theorem 4 to the sequence $(\sigma_n/\sigma_1)_{n \in \mathbb{N}}$. In this case, we

just have to replace σ_2 , C and $a_{n,d}$ by the values σ_2/σ_1 , C/σ_1 and $a_{n,d}/\sigma_1^d$ in the above estimates.

Proof. If $\sigma_2 = 1$, we have $\beta(d, n) = \alpha(d, \delta) = 0$. In that case, both estimates hold true, since $a_{n,d} = 1$ for all $n \leq (1 + v)^d$. Let hence $\sigma_2 < 1$.

Proof of (i): Let $n \in \mathbb{N}$. There is some $L > 0$ such that $a_{n,d} = \sigma_2^L$. If $\sigma_{\mathbf{n}} \geq a_{n,d}$, the number l of components of \mathbf{n} not equal to one is at most $\lfloor L \rfloor$ and hence

$$\begin{aligned} n &\leq \# \{ \mathbf{n} \in \mathbb{N}^d \mid \sigma_{\mathbf{n}} \geq a_{n,d} \} \\ &= \sum_{l=0}^{\min\{\lfloor L \rfloor, d\}} \# \{ \mathbf{n} \in \mathbb{N}^d \mid \# \{ 1 \leq j \leq d \mid \mathbf{n}_j \neq 1 \} = l \wedge \sigma_{\mathbf{n}} \geq a_{n,d} \} \\ &= 1 + \sum_{l=1}^{\min\{\lfloor L \rfloor, d\}} \binom{d}{l} \# \{ \mathbf{n} \in \{2, 3, \dots\}^l \mid \sigma_{\mathbf{n}} \geq a_{n,d} \}. \end{aligned} \quad (3.4)$$

Since $\sigma_{\mathbf{n}} \leq C^l \prod_{j=1}^l \mathbf{n}_j^{-s}$ for $\mathbf{n} \in \mathbb{N}^l$, we get with Lemma 2

$$\# \{ \mathbf{n} \in \{2, 3, \dots\}^l \mid \sigma_{\mathbf{n}} \geq a_{n,d} \} \leq A_2 \left(C^{l/s} a_{n,d}^{-1/s}, l \right) \leq C^{(1+\delta)l/s} a_{n,d}^{-(1+\delta)/s} \delta^{-l} \quad (3.5)$$

for each $l \in \{1, \dots, \min\{L, d\}\}$. Obviously,

$$1 \leq a_{n,d}^{-(1+\delta)/s} = \binom{d}{0} \cdot C^{0/s} a_{n,d}^{-(1+\delta)/s} \delta^0. \quad (3.6)$$

Inserting these bounds in (3.4) yields

$$\begin{aligned} n &\leq \sum_{l=0}^{\min\{\lfloor L \rfloor, d\}} \binom{d}{l} \cdot C^{(1+\delta)l/s} a_{n,d}^{-(1+\delta)/s} \delta^{-l} \leq a_{n,d}^{-(1+\delta)/s} \sum_{l=0}^{\min\{\lfloor L \rfloor, d\}} \frac{d^l}{l!} C^{(1+\delta)l/s} \delta^{-l} \\ &\leq \sigma_2^{-(1+\delta)L/s} d^L \sum_{l=0}^{\min\{\lfloor L \rfloor, d\}} \frac{\left(\frac{C^{(1+\delta)/s}}{\delta} \right)^l}{l!} \leq \left(\sigma_2^{-(1+\delta)/s} \cdot d \right)^L \exp \left(\frac{C^{(1+\delta)/s}}{\delta} \right) \end{aligned} \quad (3.7)$$

and hence

$$L \geq \frac{\log n - \frac{C^{(1+\delta)/s}}{\delta}}{\log \left(\sigma_2^{-(1+\delta)/s} \cdot d \right)}. \quad (3.8)$$

Thus

$$a_{n,d} = \sigma_2^L \leq \exp \left(\frac{\left(\frac{C^{(1+\delta)/s}}{\delta} - \log n \right) \log \sigma_2^{-1}}{\log \left(\sigma_2^{-(1+\delta)/s} \cdot d \right)} \right) = \left(\frac{\exp \left(\frac{C^{(1+\delta)/s}}{\delta} \right)}{n} \right)^{\alpha(d,\delta)} \quad (3.9)$$

with

$$\alpha(d, \delta) = \frac{\log \sigma_2^{-1}}{\log \left(\sigma_2^{-(1+\delta)/s} \cdot d \right)}. \quad (3.10)$$

Proof of (ii): Let $vd < n \leq (1+v)^d$. Then there is some $L \in \{1, \dots, d-1\}$ such that $a_{n,d} \in [\sigma_2^{L+1}, \sigma_2^L)$. Clearly,

$$n > \# \{ \mathbf{n} \in \mathbb{N}^d \mid \sigma_{\mathbf{n}} > a_{n,d} \} \geq \sum_{l=1}^L \binom{d}{l} \# \{ \mathbf{n} \in \{2, 3, \dots\}^l \mid \sigma_{\mathbf{n}} > a_{n,d} \}. \quad (3.11)$$

Since $\sigma_{\mathbf{n}} > a_{n,d}$ for $\mathbf{n} \in \mathbb{N}^l$, if $l \leq L$ and $\sigma_{\mathbf{n}_j} = \sigma_2$ for $j = 1 \dots l$, we obtain

$$n \geq \sum_{l=0}^L \binom{d}{l} v^l \geq \sum_{l=0}^L \binom{L}{l} \left(\frac{d}{L} \right)^l v^l = \left(1 + \frac{vd}{L} \right)^L. \quad (3.12)$$

Since d/L is bigger than one, this yields in particular that

$$L \leq \log_{1+v} n. \quad (3.13)$$

We insert this auxiliary estimate on L in (3.12) and get

$$n \geq \left(1 + \frac{vd}{\log_{1+v} n} \right)^L, \quad (3.14)$$

or equivalently

$$L \leq \frac{\log n}{\log \left(1 + \frac{vd}{\log_{1+v} n} \right)}. \quad (3.15)$$

We recall that $a_{n,d} \geq \sigma_2^{L+1}$ and realize that the proof is finished. \square

The bounds of Theorem 4 are very explicit, but complex. One might be bothered by the dependence of the exponent $\beta(d, n)$ in the lower bound on n . This can be overcome, if we restrict the lower bound to the case $n \leq (1+v)^{d^a}$ for some

$0 < a < 1$ and replace $\beta(d, n)$ by

$$\tilde{\beta}(d, n) = \frac{\log \sigma_2^{-1}}{\log(1 + v \cdot d^{1-a})}. \quad (3.16)$$

Of course, we throw away information this way. Similarly, we get a worse but still valid estimate, if we replace v by one. Note that these lower bounds are valid for any zero sequence $(\sigma_n)_{n \in \mathbb{N}}$, independent of its rate of convergence.

The constants 1 , σ_2 and $\tilde{C}(\delta)$ are independent of the power d . The additional parameter δ in the upper bound was introduced to maximize the exponent $\alpha(d, \delta)$. If δ tends to zero, $\alpha(d, \delta)$ gets bigger, but also the constant $\tilde{C}(\delta)$ explodes.

For large values of d and if n is significantly smaller than $(1+v)^d$, the exponents in both the upper and the lower bound are approximately $\frac{\log \sigma_2^{-1}}{\log d}$. For arbitrary values of σ_1 , both exponents are close to $t_d = \frac{\log(\sigma_2/\sigma_1)^{-1}}{\log d}$. In other words, the sequence $(a_{n,d})_{n \in \mathbb{N}}$ preasymptotically roughly decays like n^{-t_d} .

Similar estimates for the case of $(a_{n,d})_{n \in \mathbb{N}}$ being the sequence of approximation numbers of the embedding $H_{\text{mix}}^s(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d)$ are established in Theorem 4.9, 4.10, 4.17 and 4.20 of [KSU15]. I will comment on differences in Section 4.

These kind of estimates are also closely related to those in [GW11, Section 3]. Using the language of generalized tractability, Gnewuch and Woźniakowski show that the supremum of all $\tau > 0$ such that there is a constant $C > 0$ with

$$a_{n,d} \leq e \left(\frac{C}{n} \right)^{\frac{\tau}{1+\log d}} \quad (3.17)$$

for all $n \in \mathbb{N}$ and $d \in \mathbb{N}$ is $\min \{s, \log \sigma_2^{-1}\}$.

4 Applications to some Tensor Power Operators

Let X and Y be Banach spaces and $T : X \rightarrow Y$ be a bounded linear operator. The n th approximation number $a_n(T)$ describes the power of the approximation of T in $\mathcal{L}(X, Y)$ by operators of rank less than n :

$$a_n(T) = \inf_{\text{rank}(A) < n} \|T - A\|. \quad (4.1)$$

In the following, X and Y are separable Hilbert spaces and T is a compact operator with infinite rank. Let us first recall some facts about such operators.

The operator $W = T^*T : X \rightarrow X$ is positive semi-definite and compact and

admits a countable orthogonal basis $(b_k)_{k \in I}$ of $N(T)^\perp$ consisting of eigenvectors. It is unique up to the choice of orthogonal bases in the finite-dimensional eigenspaces of W and can be characterized as the orthogonal basis $(b_k)_{k \in I}$ of $N(T)^\perp$ for which $(Tb_k)_{k \in I}$ is an orthogonal basis of $\overline{R(T)}$. The operator T has the representation

$$Tf = \sum_{k \in I} \langle f, b_k \rangle_X \cdot Tb_k \quad \text{for } f \in X. \quad (4.2)$$

This is often referred to as the singular value decomposition of T . The eigenvalues of W take the form

$$\lambda_k = \frac{\langle Wb_k, b_k \rangle_X}{\langle b_k, b_k \rangle_X} = \frac{\|Tb_k\|_Y^2}{\|b_k\|_X^2} \quad (4.3)$$

for $k \in I$ and can be rearranged to a nonincreasing zero sequence $(\lambda_{k_n})_{n \in \mathbb{N}}$. It is not hard to see that the operator

$$A_n : X \rightarrow Y, \quad Af = \sum_{j=1}^{n-1} \langle f, b_{k_j} \rangle_X \cdot Tb_{k_j}, \quad (4.4)$$

is the best possible approximation of T by an operator of rank less than n and its n th approximation number σ_n is given by

$$a_n(T) = \|T - A_n\| = \frac{\|Tb_{k_n}\|_Y}{\|b_{k_n}\|_X} = \min_{\substack{V \subseteq X \\ \dim(V)=n-1}} \max_{\substack{f \perp V \\ \|f\|_X=1}} \|Tf\|_Y. \quad (4.5)$$

In particular, the norm of T is σ_1 and the n th approximation number of T is the square root of the n th largest eigenvalue of W , which tends to zero as n tends to infinity. In this setting, σ_n coincides with all other s -numbers, like Gelfand or Kolmogorov numbers. It is also often referred to as n th singular value of T .

Suppose now that X and Y are Hilbert spaces of real or complex-valued functions on the compact manifold G . Their d -th tensor powers are the complete d -fold tensor product Hilbert spaces

$$X_d = X \otimes \dots \otimes X, \quad Y_d = Y \otimes \dots \otimes Y, \quad (4.6)$$

consisting of real or complex-valued functions on the product manifold G^d . The d -th tensor power of the operator T is the tensor product operator

$$T_d = T \otimes \dots \otimes T : X_d \rightarrow Y_d, \quad (4.7)$$

which is itself compact. The family $(b_{\mathbf{k}})_{\mathbf{k} \in I^d}$ of tensor product functions

$$b_{\mathbf{k}} = b_{\mathbf{k}_1} \otimes \dots \otimes b_{\mathbf{k}_d}, \quad (4.8)$$

is an orthogonal basis of $N(T_d)^\perp$ consisting of eigenvectors of $W_d = T_d^* T_d$ for the eigenvalues $\lambda_{\mathbf{k}} = \prod_{j=1}^d \lambda_{k_j}$. Like before, the n th approximation number $a_{n,d} = a_n(T_d)$ of T_d is the square root of the n th biggest eigenvalue of W_d . Hence, the sequence $(a_{n,d})_{n \in \mathbb{N}}$ of approximation numbers of T_d is nothing else than the nonincreasing rearrangement of the d -th tensor power of the sequence $(\sigma_n)_{n \in \mathbb{N}}$ of approximation numbers of T .

4.1 Approximation of Mixed Order Sobolev Functions on the Torus

Let \mathbb{T} be the 1-torus, the circle, represented by the interval $[a, b]$, where the two end points $a < b$ are identified. By $L_2(\mathbb{T})$, we denote the Hilbert space of square-integrable functions on \mathbb{T} , equipped with the scalar product

$$\langle f, g \rangle = \frac{1}{L} \int_{\mathbb{T}} f(x) \overline{g(x)} \, dx \quad (4.9)$$

and the induced norm $\|\cdot\|$ for some $L > 0$. Typical normalizations are $[a, b] \in \{[0, 1], [-1, 1], [0, 2\pi]\}$ and $L \in \{1, b - a\}$. The family $(b_k)_{k \in \mathbb{Z}}$ with

$$b_k(x) = \sqrt{\frac{L}{b-a}} \exp\left(2\pi i k \frac{x-a}{b-a}\right) \quad (4.10)$$

is an orthonormal basis of $L_2(\mathbb{T})$, its Fourier basis, and

$$\hat{f}(k) = \langle f, b_k \rangle \quad (4.11)$$

is the k th Fourier coefficient of $f \in L_2(\mathbb{T})$. By Parseval's identity,

$$\|f\|^2 = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 \quad \text{and} \quad \langle f, g \rangle = \sum_{k \in \mathbb{Z}} \hat{f}(k) \cdot \overline{\hat{g}(k)}. \quad (4.12)$$

Let $w = (w_k)_{k \in \mathbb{N}}$ be a nondecreasing sequence of real numbers with $w_0 = 1$ and let $w_{-k} = w_k$ for $k \in \mathbb{N}$ and so let \tilde{w} . The univariate Sobolev space $H^w(\mathbb{T})$ is the Hilbert space of functions $f \in L_2(\mathbb{T})$ for which

$$\|f\|_w^2 = \sum_{k \in \mathbb{Z}} w_k^2 \cdot |\hat{f}(k)|^2 \quad (4.13)$$

is finite, equipped with the scalar product

$$\langle f, g \rangle_w = \sum_{k \in \mathbb{Z}} w_k \hat{f}(k) \cdot \overline{w_k \hat{g}(k)}. \quad (4.14)$$

Note that $H^w(\mathbb{T})$ and $H^{\tilde{w}}(\mathbb{T})$ coincide and their norms are equivalent, if and only if $w \sim \tilde{w}$. In case $w_k \sim k^s$ for some $s \geq 0$, the space $H^w(\mathbb{T})$ is the classical Sobolev space of periodic univariate functions with fractional smoothness s , also denoted by $H^s(\mathbb{T})$. In particular, $H^w(\mathbb{T}) = L_2(\mathbb{T})$ for $w \equiv 1$.

In accordance with previous notation, let $X = H^w(\mathbb{T})$ and $Y = H^{\tilde{w}}(\mathbb{T})$. The embedding T of X into Y is compact, if and only if w_k/\tilde{w}_k tends to infinity as k tends to infinity. The Fourier basis $(b_k)_{k \in \mathbb{Z}}$ is an orthogonal basis of X consisting of eigenfunctions of $W = T^*T$ with corresponding eigenvalues

$$\lambda_k = \frac{\|b_k\|_Y^2}{\|b_k\|_X^2} = \frac{\tilde{w}_k^2}{w_k^2}. \quad (4.15)$$

The n th approximation number σ_n of this embedding is the square root of the n th biggest eigenvalue. Replacing the Fourier weight sequences w and \tilde{w} by equivalent sequences does hence not affect the order of convergence of the corresponding approximation numbers, but it may drastically affect their asymptotic constants and preasymptotic behavior. If $Y = L_2(\mathbb{T})$, we obtain

$$\sigma_n = w_{k_n}^{-1}, \quad \text{where } k_n = (-1)^n \lfloor n/2 \rfloor. \quad (4.16)$$

Note that σ_1 , the norm of the embedding T , is always one.

The d -th tensor power $X_d = H_{\text{mix}}^w(\mathbb{T}^d)$ of X is a space of mixed order Sobolev functions on the d -torus. If $w_k \sim k^s$ for some $s \geq 0$, this is the space $H_{\text{mix}}^s(\mathbb{T}^d)$ of functions with dominating mixed smoothness s . If even $s \in \mathbb{N}_0$, this space consists of all real-valued functions on the d -torus, which have a weak (or distributional) derivative of order α in $L_2(\mathbb{T}^d)$ for any $\alpha \in \{0, 1, \dots, s\}^d$. Of course, the same holds for the d -th tensor power $Y_d = H_{\text{mix}}^{\tilde{w}}(\mathbb{T}^d)$ of Y . The tensor power operator $T_d : X_d \rightarrow Y_d$ is the compact embedding of $H_{\text{mix}}^w(\mathbb{T}^d)$ into $H_{\text{mix}}^{\tilde{w}}(\mathbb{T}^d)$. Hence, the approximation numbers $(a_{n,d})_{n \in \mathbb{N}}$ of this embedding are the nonincreasing rearrangement of the d -th tensor power of $(\sigma_n)_{n \in \mathbb{N}}$.

If $(\tilde{w}_k/w_k)_{k \in \mathbb{N}}$ is of polynomial decay, Theorem 3 and Theorem 4 apply. We formulate the results for the embedding of $H_{\text{mix}}^s(\mathbb{T}^d)$ into $L_2(\mathbb{T}^d)$, where $H_{\text{mix}}^s(\mathbb{T}^d)$

will be equipped with different equivalent norms, indicated by the notation

$$\begin{aligned}
H_{\text{mix}}^{s,\circ,\gamma}(\mathbb{T}^d), \quad & \text{if } w_k = \left(\sum_{l=0}^s \left| \gamma^{-1} \frac{2\pi k}{b-a} \right|^{2l} \right)^{1/2}, \\
H_{\text{mix}}^{s,*,\gamma}(\mathbb{T}^d), \quad & \text{if } w_k = \left(1 + \left| \gamma^{-1} \frac{2\pi k}{b-a} \right|^{2s} \right)^{1/2}, \\
H_{\text{mix}}^{s,+, \gamma}(\mathbb{T}^d), \quad & \text{if } w_k = \left(1 + \left| \gamma^{-1} \frac{2\pi k}{b-a} \right|^2 \right)^{s/2}, \\
H_{\text{mix}}^{s,\#, \gamma}(\mathbb{T}^d), \quad & \text{if } w_k = \left(1 + \left| \gamma^{-1} \frac{2\pi k}{b-a} \right|^s \right),
\end{aligned} \tag{4.17}$$

for some $\gamma > 0$. The last three norms are due to Kühn, Sickel and Ullrich [KSU15], who study all these norms for $\gamma = 1$, $L = 1$ and $[a, b] = [0, 2\pi]$. The last norm is also studied by Chernov and Dũng in [CD16] for $L = 2\pi$, $[a, b] = [-\pi, \pi]$ and arbitrary values of γ . If s is a natural number, the first two scalar products take the form

$$\begin{aligned}
\langle f, g \rangle_{H_{\text{mix}}^{s,\circ,\gamma}} &= \sum_{\alpha \in \{0, \dots, s\}^d} \gamma^{-2s|\alpha|} \langle D^\alpha f, D^\alpha g \rangle, \\
\langle f, g \rangle_{H_{\text{mix}}^{s,*,\gamma}} &= \sum_{\alpha \in \{0, s\}^d} \gamma^{-2s|\alpha|} \langle D^\alpha f, D^\alpha g \rangle.
\end{aligned} \tag{4.18}$$

This is why $H_{\text{mix}}^{s,\circ,1}(\mathbb{T}^d)$ and $H_{\text{mix}}^{s,*,1}(\mathbb{T}^d)$ might be considered the most natural choice. Note that the corresponding approximation numbers of the embedding T_d are independent of the normalization constant L , but they do depend on the length of the interval $[a, b]$.

Corollary 1. *The following limits exist and coincide:*

$$\left. \begin{aligned}
& \lim_{n \rightarrow \infty} a_n \left(H_{\text{mix}}^{s,\circ,\gamma}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d) \right) \cdot n^s (\log n)^{-s(d-1)} \\
& \lim_{n \rightarrow \infty} a_n \left(H_{\text{mix}}^{s,*,\gamma}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d) \right) \cdot n^s (\log n)^{-s(d-1)} \\
& \lim_{n \rightarrow \infty} a_n \left(H_{\text{mix}}^{s,+, \gamma}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d) \right) \cdot n^s (\log n)^{-s(d-1)} \\
& \lim_{n \rightarrow \infty} a_n \left(H_{\text{mix}}^{s,\#, \gamma}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d) \right) \cdot n^s (\log n)^{-s(d-1)}
\end{aligned} \right\} = \left(\frac{\left(\gamma \frac{b-a}{\pi} \right)^d}{(d-1)!} \right)^s.$$

Of course, this coincides with the limits computed in [KSU15], if $\gamma^{-1} \frac{b-a}{\pi} = 2$. The third limit (for $[a, b] = [-\pi, \pi]$, $L = 2\pi$ and any $\gamma > 0$) may not be written down explicitly in [CD16], but can be derived from their Theorem 4.6.

Corollary 2. Let $\square \in \{\circ, *, +, \#\}$. For any $\delta \in (0, 1]$, $s > 0$, $d \in \mathbb{N}$ and $n \in \mathbb{N}$ with $2d < n \leq 3^d$,

$$\sigma_2 \cdot \left(\frac{1}{n}\right)^{\beta_{\square}(d,n)} \leq a_n \left(H_{\text{mix}}^{s,\square,\gamma}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d)\right) \leq \left(\frac{\tilde{C}(\delta)}{n}\right)^{\alpha_{\square}(d,\delta)},$$

where σ_2 is the value w_1^{-1} from above and

$$\begin{aligned} \tilde{C}(\delta) &= \exp\left(\frac{(3/\eta)^{1+\delta}}{\delta}\right), & \eta &= \gamma^{-1} \frac{2\pi}{b-a}, \\ \alpha_{\circ}(d, \delta) &= \frac{\frac{1}{2} \log(\sum_{l=0}^s \eta^{2l})}{\log d + \frac{1+\delta}{2s} \cdot \log(\sum_{l=0}^s \eta^{2l})}, & \beta_{\circ}(d, n) &= \frac{\frac{1}{2} \log(\sum_{l=0}^s \eta^{2l})}{\log\left(1 + \frac{2}{\log_3 n} d\right)}, \\ \alpha_{*}(d, \delta) &= \frac{\frac{1}{2} \log(1 + \eta^{2s})}{\log d + \frac{1+\delta}{2s} \cdot \log(1 + \eta^{2s})}, & \beta_{*}(d, n) &= \frac{\frac{1}{2} \log(1 + \eta^{2s})}{\log\left(1 + \frac{2}{\log_3 n} d\right)}, \\ \alpha_{+}(d, \delta) &= \frac{\frac{s}{2} \log(1 + \eta^2)}{\log d + \frac{1+\delta}{2} \cdot \log(1 + \eta^2)}, & \beta_{+}(d, n) &= \frac{\frac{s}{2} \log(1 + \eta^2)}{\log\left(1 + \frac{2}{\log_3 n} d\right)}, \\ \alpha_{\#}(d, \delta) &= \frac{s \log(1 + \eta)}{\log d + (1 + \delta) \log(1 + \eta)}, & \beta_{\#}(d, n) &= \frac{s \log(1 + \eta)}{\log\left(1 + \frac{2}{\log_3 n} d\right)}. \end{aligned}$$

The upper bound even holds for all $n \geq 2$.

Let us consider the setting of [KSU15], where $\gamma = 1$ and $b - a = 2\pi$ and hence η is one. The exponents $\alpha_{\#}(d, \delta) = \frac{s}{\log_2 d + 1 + \delta}$ and $\alpha_{+}(d, \delta) = \frac{s}{2 \log_2 d + 1 + \delta}$ in our upper bounds are slightly better than the exponents $\frac{s}{\log_2 d + 2}$ and $\frac{s}{2 \log_2 d + 4}$ in Theorem 4.9, 4.10 and Theorem 4.17 of [KSU15], but almost the same. Also the lower bounds basically coincide. Regarding $H_{\text{mix}}^{s,*,1}(\mathbb{T}^d)$, Kühn, Sickel and Ullrich only studied the case $1/2 \leq s \leq 1$ in Theorem 4.20. As we see now, there is a major difference between this natural norm and the last two norms: For large dimensions d , the preasymptotic behavior of the approximation numbers is roughly $n^{-t_{d,\square}}$, where

$$t_{d,\circ} = \frac{\log(s+1)}{2 \log d}, \quad t_{d,*} = \frac{1}{2 \log_2 d}, \quad t_{d,+} = \frac{s}{2 \log_2 d}, \quad t_{d,\#} = \frac{s}{\log_2 d}. \quad (4.19)$$

This means that the smoothness of the space only has a minor or even no impact on the preasymptotic decay of the approximation numbers, if $H_{\text{mix}}^s(\mathbb{T}^d)$ is equipped with one of the natural norms $\|\cdot\|_{H_{\text{mix}}^{s,\circ,1}}$ or $\|\cdot\|_{H_{\text{mix}}^{s,*,1}}$.

This changes, however, if the value of $\eta = \frac{2\pi}{\gamma(b-a)}$ changes. If η is larger than

one, because we consider a shorter interval $[a, b]$ or because we put some weight $\gamma < \frac{2\pi}{b-a}$, also the exponents $t_{d,o}$ and $t_{d,*}$ get linear in s . For the other two families of norms, the smoothness does show and the value of η is less important.

There are no preasymptotic estimates in [CD16].

4.2 Approximation of Mixed Order Jacobi Functions on the Cube

The above results also apply to the approximation numbers of the embedding of mixed order Jacobi functions on the d -cube in the corresponding L_2 -space as considered in [CD16, Section 5].

Let \mathbb{I} be the 1-cube, a line segment, represented by $[-1, 1]$. For fixed parameters $\alpha, \beta > -1$ with $a := \frac{\alpha+\beta+1}{2} > 0$, the weighted L_2 -space $Y = L_2(\mathbb{I}, w)$ is the Hilbert space of measurable, real-valued functions on \mathbb{I} with

$$\int_{\mathbb{I}} f(x)^2 w(x) \, dx < \infty, \quad (4.20)$$

equipped with the scalar product

$$\langle f, g \rangle = \int_{\mathbb{I}} f(x)g(x)w(x) \, dx \quad (4.21)$$

and the induced norm $\|\cdot\|$, where $w : \mathbb{I} \rightarrow \mathbb{R}$ is the Jacobi weight

$$w(x) = (1-x)^\alpha (1+x)^\beta. \quad (4.22)$$

This reduces to the classical space of square-integrable functions, if both parameters are zero. As α respectively β increases, the space grows, since we allow for stronger singularities on the right respectively left boundary of the interval, and vice versa.

The family of Jacobi polynomials $(P_k)_{k \in \mathbb{N}_0}$ is an orthogonal basis of Y . These polynomials can be defined as the unique solutions of the differential equations

$$\mathcal{L}P_k = k(k+2a)P_k \quad (4.23)$$

for the second order differential operator

$$\mathcal{L} = -w(x)^{-1} \frac{d}{dx} \left((1-x^2) w(x) \frac{d}{dx} \right) \quad (4.24)$$

that satisfy

$$P_k(1) = \binom{k+\alpha}{k} \quad \text{and} \quad P_k(-1) = (-1)^k \binom{k+\beta}{k}. \quad (4.25)$$

We denote the k th Fourier coefficient of f with respect to the normalized Jacobi basis by f_k . The scalar product in Y hence admits the representation

$$\langle f, g \rangle = \sum_{k=0}^{\infty} f_k g_k. \quad (4.26)$$

For $s > 0$ let $X = K^s(\mathbb{I}, w)$ be the Hilbert space of functions $f \in Y$ with

$$\sum_{k=0}^{\infty} (1 + a^{-1}k)^{2s} f_k^2 < \infty, \quad (4.27)$$

equipped with the scalar product

$$\langle f, g \rangle_s = \sum_{k=0}^{\infty} (1 + a^{-1}k)^{2s} f_k g_k \quad (4.28)$$

and the induced norm $\|\cdot\|_s$. Obviously, $(P_k)_{k \in \mathbb{N}_0}$ is an orthogonal basis of X , too. In case s is an even integer, this is the space of all functions $f \in L_2(\mathbb{I}, w)$ such that $\mathcal{L}^j f \in L_2(\mathbb{I}, w)$ for $j = 1 \dots \frac{s}{2}$ and the scalar product

$$\langle f, g \rangle_{s,*} = \sum_{j=0}^{s/2} \langle \mathcal{L}^j f, \mathcal{L}^j g \rangle \quad (4.29)$$

is equivalent to the one above. The parameter s can hence be interpreted as smoothness of the functions in $K^s(\mathbb{I}, w)$. The embedding T of X into Y is compact and its n th approximation number is given by

$$\sigma_n = a_n(T) = \frac{\|P_{n-1}\|}{\|P_{n-1}\|_s} = (1 + a^{-1}(n-1))^{-s}. \quad (4.30)$$

We can apply our theorems to study the approximation numbers of the d -th tensor power T_d of T . This is the embedding of $X_d = K^s(\mathbb{I}^d, w_d)$ into $Y_d = L_2(\mathbb{I}^d, w_d)$, where Y_d is the weighted L_2 -space on the d -cube with respect to the Jacobi weight $w_d = w \otimes \dots \otimes w$ and X_d is the subspace of Jacobi functions of mixed order s . Like in the univariate case, X_d can be described via differentials of

dominating mixed order s and less, if s is an even integer.

Corollary 3. *For any $d \in \mathbb{N}$ and $s > 0$, the following limit exists:*

$$\lim_{n \rightarrow \infty} a_n \left(K^s (\mathbb{I}^d, w_d) \hookrightarrow L_2 (\mathbb{I}^d, w_d) \right) \cdot n^s (\log n)^{-s(d-1)} = \left(\frac{a^d}{(d-1)!} \right)^s.$$

This result could also be derived from Theorem 5.5 in [CD16]. In addition, we get the following preasymptotic estimates:

Corollary 4. *For any $\delta \in (0, 1]$, $s > 0$, $d \in \mathbb{N}$ and $n \in \mathbb{N}$ with $d < n \leq 2^d$,*

$$\left(\frac{a}{a+1} \right)^s \left(\frac{1}{n} \right)^{p_{s,a,d,n}} \leq a_n \left(K^s (\mathbb{I}^d, w_d) \hookrightarrow L_2 (\mathbb{I}^d, w_d) \right) \leq \left(\frac{\exp \left(\frac{(2a)^{1+\delta}}{\delta} \right)}{n} \right)^{q_{s,a,d,\delta}}$$

$$\text{with } p_{s,a,d,n} = \frac{s \log \frac{a+1}{a}}{\log \left(1 + \frac{d}{\log_2 n} \right)} \quad \text{and} \quad q_{s,a,d,\delta} = \frac{s \log \frac{a+1}{a}}{\log d + (1+\delta) \log \frac{a+1}{a}}.$$

The upper bound even holds for all $n \geq 2$.

This means that for large dimension d , a preasymptotic decay of approximate order $t_d = s \log \frac{a+1}{a} / \log d$ in n can be observed.

4.3 Approximation of Mixed Order Sobolev Functions on the Cube

Another example of a tensor power operator is given by the L_2 -embedding of mixed order Sobolev functions on the d -cube. Let \mathbb{I} be the 1-cube and \mathbb{T} be the 1-torus. Both shall be represented by the interval $[a, b]$, where a and b are identified in the second case. For any $s \in \mathbb{N}_0$, the vector space

$$H^s (\mathbb{I}) = \left\{ f \in L_2 (\mathbb{I}) \mid f^{(l)} \in L_2 (\mathbb{I}) \text{ for } 1 \leq l \leq s \right\}, \quad (4.31)$$

equipped with the scalar product

$$\langle f, g \rangle_s = \sum_{l=0}^s \int_a^b f^{(l)}(x) \cdot \overline{g^{(l)}(x)} \, dx \quad (4.32)$$

and induced norm $\|\cdot\|_s$, is a Hilbert space, the Sobolev space of order s on \mathbb{I} . In case $s = 0$, it coincides with $L_2 (\mathbb{I})$. The subset

$$H^s (\mathbb{T}) = \left\{ f \in H^s (\mathbb{I}) \mid f^{(l)}(a) = f^{(l)}(b) \text{ for } l = 0, 1, \dots, s-1 \right\} \quad (4.33)$$

of periodic functions is a closed subspace with codimension s , the Sobolev space of order s on \mathbb{T} . By means of Parseval's identity and integration by parts, the above norm can be rearranged to

$$\|f\|_s^2 = \sum_{k \in \mathbb{Z}} \left| \hat{f}(k) \right|^2 \sum_{l=0}^s \left| \frac{2\pi k}{b-a} \right|^{2l} \quad \text{for } f \in H^s(\mathbb{T}), \quad (4.34)$$

where

$$\hat{f}(k) = \sqrt{\frac{1}{b-a}} \int_a^b f(x) \cdot \exp\left(-2\pi i k \frac{x-a}{b-a}\right) dx \quad (4.35)$$

is the k th Fourier coefficient of f . In the limiting case $s = \infty$, the Sobolev space $H^\infty(\mathbb{I})$ shall be defined as the Hilbert space

$$H^s(\mathbb{I}) = \left\{ f \in \mathcal{C}^\infty(\mathbb{I}) \mid \sum_{l=0}^{\infty} \|f^{(l)}\|_0^2 < \infty \right\}, \quad (4.36)$$

equipped with the scalar product (4.32) for $s = \infty$. It contains all polynomials and is hence infinite-dimensional. The space $H^\infty(\mathbb{T})$ shall be the closed subspace of periodic functions, i.e.

$$H^\infty(\mathbb{T}) = \{f \in H^\infty(\mathbb{I}) \mid f^{(l)}(a) = f^{(l)}(b) \text{ for any } l \in \mathbb{N}_0\}. \quad (4.37)$$

Note that (4.34) also holds for $s = \infty$. Hence,

$$H^\infty(\mathbb{T}) = \text{span} \left\{ \exp\left(2\pi i k \frac{\cdot - a}{b-a}\right) \mid k \in \mathbb{Z} \text{ with } \left| \frac{2\pi k}{b-a} \right| < 1 \right\} \quad (4.38)$$

is finite-dimensional with dimension $2\lceil \frac{b-a}{2\pi} \rceil - 1$. In case $b-a \leq 2\pi$, it consists of constant functions only.

If s is positive, $H^s(\mathbb{I})$ is compactly embedded into $L_2(\mathbb{I})$. Let $\sigma_n(s)$ be the n th singular value of this embedding and let $\tilde{\sigma}_n(s)$ be the n th singular value of the embedding of the subspace $H^s(\mathbb{T})$ into $L_2(\mathbb{T})$. We want to study the approximation numbers $(a_{n,d}(s))_{n \in \mathbb{N}}$ of the compact embedding of the d -th tensor power space $H_{\text{mix}}^s(\mathbb{I}^d)$ into $L_2(\mathbb{I}^d)$. If s is finite, this is the space

$$H_{\text{mix}}^s(\mathbb{I}^d) = \left\{ f \in L_2(\mathbb{I}^d) \mid D^\alpha f \in L_2(\mathbb{I}^d) \text{ for each } \alpha \in \{0, \dots, s\}^d \right\}, \quad (4.39)$$

equipped with the scalar product

$$\langle f, g \rangle_s = \sum_{\alpha \in \{0, \dots, s\}^d} \int_{[a, b]^d} D^\alpha f(\mathbf{x}) \cdot \overline{D^\alpha g(\mathbf{x})} \, d\mathbf{x}. \quad (4.40)$$

See Section 4.1 for a treatment of the L_2 -approximation numbers of the d -th tensor power $H_{\text{mix}}^s(\mathbb{T}^d)$ of the periodic space.

By means of Theorem 3 and Theorem 4, it is enough to study the singular values $\sigma_n(s)$ of the embedding in the univariate case. As we have seen in Section 4.1,

$$\tilde{\sigma}_n(s) = \left(\sum_{l=0}^s \left| \frac{2\pi \lfloor n/2 \rfloor}{b-a} \right|^{2l} \right)^{-1/2} \quad \text{for } n \in \mathbb{N} \text{ and } s \in \mathbb{N} \quad (4.41)$$

and in particular,

$$\lim_{n \rightarrow \infty} n^s \tilde{\sigma}_n(s) = \left(\frac{b-a}{\pi} \right)^s. \quad (4.42)$$

The singular values for nonperiodic functions, on the other hand, are not known explicitly. However, $\sigma_n(s)$ and $\tilde{\sigma}_n(s)$ interrelate as follows.

Lemma 3. *For any $n \in \mathbb{N}$ and $s \in \mathbb{N}$, it holds that $\sigma_{n+s}(s) \leq \tilde{\sigma}_n(s) \leq \sigma_n(s)$.*

Proof. The second inequality is obvious, since $H^s(\mathbb{T})$ is a subspace of $H^s(\mathbb{I})$. The first inequality is true, since the codimension of this subspace is s . Let U be the orthogonal complement of $H^s(\mathbb{T})$ in $H^s(\mathbb{I})$. By relation 4.5,

$$\begin{aligned} \sigma_{n+s}(s) &= \min_{\substack{V \subseteq H^s(\mathbb{I}) \\ \dim(V)=n+s-1}} \max_{\substack{f \perp V \\ \|f\|_s=1}} \|f\|_0 \leq \min_{\substack{\tilde{V} \subseteq H^s(\mathbb{T}) \\ \dim(V)=n-1}} \max_{\substack{f \perp (\tilde{V} \oplus U) \\ \|f\|_s=1}} \|f\|_0 \\ &= \min_{\substack{\tilde{V} \subseteq H^s(\mathbb{T}) \\ \dim(V)=n-1}} \max_{\substack{f \in H^s(\mathbb{T}), f \perp \tilde{V} \\ \|f\|_s=1}} \|f\|_0 = \tilde{\sigma}_n(s). \end{aligned} \quad (4.43)$$

Note that the same argument is not valid for $d > 1$. In this case, the codimension of $H_{\text{mix}}^s(\mathbb{T}^d)$ in $H_{\text{mix}}^s(\mathbb{I}^d)$ is not finite. \square

Lemma 3 implies that the asymptotic constants of the approximation numbers for the periodic and the nonperiodic functions coincide in the univariate case:

$$\begin{aligned} \lim_{n \rightarrow \infty} n^s \tilde{\sigma}_n(s) &\leq \lim_{n \rightarrow \infty} n^s \sigma_n(s) = \lim_{n \rightarrow \infty} (n+s)^s \sigma_{n+s}(s) \\ &= \lim_{n \rightarrow \infty} n^s \sigma_{n+s}(s) \leq \lim_{n \rightarrow \infty} n^s \tilde{\sigma}_n(s). \end{aligned} \quad (4.44)$$

Theorem 1 implies that they also coincide in the multivariate case.

Corollary 5. *For any $d \in \mathbb{N}$ and $s \in \mathbb{N}$, the following limit exists:*

$$\lim_{n \rightarrow \infty} a_n \left(H_{\text{mix}}^s(\mathbb{I}^d) \hookrightarrow L_2(\mathbb{I}^d) \right) \cdot n^s (\log n)^{-s(d-1)} = \left(\frac{(b-a)^d}{\pi^d (d-1)!} \right)^s.$$

As depicted in Section 3, the approximation numbers $a_{n,d}(s)$ show a preasymptotic decay of approximate order $\frac{\log \sigma_2^{-1}(s)}{\log d}$. Lemma 3 gives no information on $\sigma_2(s)$. However, relation (4.5) implies that

$$\sigma_2(\infty) = \max_{f \perp 1, f \neq 0} \frac{\|f\|_0}{\|f\|_\infty} \geq \frac{\|2x - a - b\|_0}{\|2x - a - b\|_\infty} = \sqrt{\frac{(b-a)^2}{12 + (b-a)^2}}. \quad (4.45)$$

If, for example, the length of the interval \mathbb{I} is one, we obtain

$$\sigma_2(\infty) \geq 0.27735. \quad (4.46)$$

Since any lower bound on $a_{n,d}(\infty)$ is a lower bound on $a_{n,d}(s)$ for $s \in \mathbb{N}$, Theorem 4 yields the following corollary.

Corollary 6. *For any $d \in \mathbb{N}$, any $s \in \mathbb{N} \cup \{\infty\}$ and $d < n \leq 2^d$,*

$$a_n \left(H_{\text{mix}}^s([0, 1]^d) \hookrightarrow L_2([0, 1]^d) \right) \geq 0.27 \cdot n^{-c(d,n)},$$

where
$$c(d, n) = \frac{1.2825}{\log \left(1 + \frac{2d}{\log_2 n} \right)} \leq 1.17.$$

On the other hand, any upper bound on $a_{n,d}(1)$ is an upper bound on $a_{n,d}(s)$ for $s \geq 1$. The singular values $\sigma_n(s)$ for $s = 1$ are known. Let T_s be the compact embedding of $H^s(\mathbb{I})$ into $L_2(\mathbb{I})$ and let $W_s = T_s^* T_s$. Then $\sigma_n(s)$ is the square-root of the n th largest eigenvalue of W_s . It is shown in [CT96] that the family $(b_k)_{k \in \mathbb{N}_0}$ is a complete orthogonal system in $H^1(\mathbb{I})$, where the function $b_k : \mathbb{I} \rightarrow \mathbb{R}$ with

$$b_k(x) = \cos \left(k\pi \cdot \frac{x-a}{b-a} \right) \quad \text{for } k \in \mathbb{N}_0 \quad (4.47)$$

is an eigenfunction of W_1 with respective eigenvalue

$$\lambda_k = \left(1 + \left(\frac{k\pi}{b-a} \right)^2 \right)^{-1}. \quad (4.48)$$

In case $\mathbb{I} = [0, 1]$, we obtain

$$\sigma_2(1) = \left(\sqrt{1 + \pi^2} \right)^{-1} \leq 0.30332 \quad (4.49)$$

and

$$\sigma_n(1) \leq 0.607 \cdot n^{-1} \quad (4.50)$$

for $n \geq 2$. Theorem 4 for $\delta = 0.65$ yields the following upper bound.

Corollary 7. *For any $d \in \mathbb{N}$, any $s \in \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}$,*

$$a_n \left(H_{\min}^s([0, 1]^d) \hookrightarrow L_2([0, 1]^d) \right) \leq \left(\frac{2}{n} \right)^{c(d)} \quad \text{with} \quad c(d) = \frac{1.1929}{2 + \log d}.$$

Apparently, the upper bound for $s = 1$ and the lower bound for $s = \infty$ are already close. The gap between the cases $s = 2$ and $s = \infty$ is even smaller.

Let c be the midpoint of \mathbb{I} and let l be its radius. Moreover, let $\hat{\omega} = \sqrt{1 + \omega^2}$ for $\omega \in \mathbb{R}$ and consider the countable sets

$$\begin{aligned} I_1 &= \left\{ \omega \geq 0 \mid \hat{\omega}^3 \cosh(\hat{\omega}l) \sin(\omega l) + \omega^3 \sinh(\hat{\omega}l) \cos(\omega l) = 0 \right\}, \\ I_2 &= \left\{ \omega > 0 \mid \hat{\omega}^3 \sinh(\hat{\omega}l) \cos(\omega l) - \omega^3 \cosh(\hat{\omega}l) \sin(\omega l) = 0 \right\}. \end{aligned} \quad (4.51)$$

It can be shown (with some effort) that the family $(b_\omega)_{\omega \in I_1 \cup I_2}$ is a complete orthogonal system in $H^2(\mathbb{I})$, where the function $b_\omega : \mathbb{I} \rightarrow \mathbb{R}$ with

$$\begin{aligned} b_\omega(x) &= \omega^2 \cdot \frac{\cosh(\hat{\omega}(x - c))}{\cosh(\hat{\omega}l)} + \hat{\omega}^2 \cdot \frac{\cos(\omega(x - c))}{\cos(\omega l)}, \quad \text{if } \omega \in I_1, \\ b_\omega(x) &= \omega^2 \cdot \frac{\sinh(\hat{\omega}(x - c))}{\sinh(\hat{\omega}l)} + \hat{\omega}^2 \cdot \frac{\sin(\omega(x - c))}{\sin(\omega l)}, \quad \text{if } \omega \in I_2, \end{aligned} \quad (4.52)$$

is an eigenfunction of W_2 with respective eigenvalue

$$\lambda_\omega = \left(1 + \omega^2 + \omega^4 \right)^{-1}. \quad (4.53)$$

In particular,

$$\sigma_2(2) = \left(\sqrt{1 + \omega_0^2 + \omega_0^4} \right)^{-1}, \quad (4.54)$$

where ω_0 is the smallest nonzero element of $I_1 \cup I_2$. If, for example, the interval \mathbb{I} has unit length, we obtain

$$\sigma_2(2) \leq 0.27795 \quad (4.55)$$

and like before,

$$\sigma_n(2) \leq 0.607 \cdot n^{-1} \quad (4.56)$$

for $n \geq 2$. Theorem 4 for $\delta = 0.65$ yields the following upper bound.

Corollary 8. *For any $d \in \mathbb{N}$, any $s \in \mathbb{N} \cup \{\infty\}$ with $s \geq 2$ and $n \in \mathbb{N}$,*

$$a_n \left(H_{\text{mix}}^s([0, 1]^d) \hookrightarrow L_2([0, 1]^d) \right) \leq \left(\frac{2}{n} \right)^{c(d)} \quad \text{with} \quad c(d) = \frac{1.2803}{2 + \log d}.$$

In short, the preasymptotic rate of the L_2 -approximation numbers of mixed order s Sobolev functions on the unit cube is $\frac{1.1929}{\log d}$ for $s = 1$, and in between $\frac{1.2803}{\log d}$ and $\frac{1.2825}{\log d}$ for any other $s \in \mathbb{N} \cup \{\infty\}$.

5 Tractability through Decreasing Complexity of the Univariate Problem

For every $d \in \mathbb{N}$, let X_d and Y_d be normed spaces and let F_d be a subset of X_d . We want to approximate the operator $T^{(d)} : F_d \rightarrow Y_d$ by an algorithm $A_n^{(d)} : F_d \rightarrow Y_d$ that uses at most n linear and continuous functionals on X_d . The n th minimal worst case error

$$e(n, d) = \inf_{A_n} \sup_{f \in F_d} \|T_d f - A_n f\|_{Y_d} \quad (5.1)$$

measures the worst case error of the best such algorithm A_n . If F_d is the unit ball of a pre-Hilbert space and $T^{(d)}$ is linear, it is known to coincide with the $(n+1)$ th approximation number of $T^{(d)}$. Conversely, the information complexity

$$n(\varepsilon, d) = \min \{n \in \mathbb{N}_0 \mid e(n, d) < \varepsilon\} \quad (5.2)$$

is the minimal number of linear and continuous functionals that is needed to achieve an error less than ε . The problem $\{T^{(d)}\}$ is called polynomially tractable, if there are nonnegative numbers C , p and q such that

$$n(\varepsilon, d) \leq C \varepsilon^{-q} d^p \quad \text{for all } d \in \mathbb{N} \text{ and } \varepsilon > 0. \quad (5.3)$$

It is called strongly polynomially tractable, if (5.3) holds with p equal to zero. See [NW08] for a detailed treatment of these and other concepts of tractability.

In the following, X_d and Y_d will be Hilbert spaces and $T^{(d)}$ will be a linear and compact norm-one operator with approximation numbers of polynomial decay.

For example, one can think of $T^{(d)}$ as the embedding of the Sobolev space $H^{s_d}(G)$ into $H^{r_d}(G)$ for some $r_d < s_d$ and a compact manifold G . Let T_d be the d -th tensor power of $T^{(d)}$. In the chosen example, this is the embedding of $H_{\text{mix}}^{s_d}(G^d)$ into $H_{\text{mix}}^{r_d}(G^d)$. We will refer to $\{T^{(d)}\}$ as the univariate and to $\{T_d\}$ as the multivariate problem. It is proven in [NW08, Theorem 5.5] that the multivariate problem is not polynomially tractable, if $T^{(d)}$ is the same operator for every $d \in \mathbb{N}$. This corresponds to the case, where the complexity of the univariate problem is constant in d . Can we achieve polynomial tractability of the multivariate problem, if the complexity of the univariate problem decreases, as d increases? If yes, to which extent do we have to simplify the univariate problem? The answer is given by the following theorem.

Theorem 5. *For every natural number d , let $T^{(d)}$ be a compact norm-one operator between Hilbert spaces and let T_d be its d -th tensor power. Assume that $a_n(T^{(d)})$ is nonincreasing in d and $a_n(T^{(1)})$ decays polynomially in n . The problem $\{T_d\}$ is strongly polynomially tractable, iff it is polynomially tractable, iff $a_2(T^{(d)})$ decays polynomially in d .*

Proof. Clearly, strong polynomial tractability implies polynomial tractability.

Let $\{T_d\}$ be polynomially tractable. Let $\sigma_{n,d}$ denote the n th approximation number of $T^{(d)}$ and $a_{n,d}$ denote the n th approximation number of T_d . There hence are nonnegative numbers C, p and q such that

$$n(\varepsilon, d) = \#\{n \in \mathbb{N} \mid a_{n,d} \geq \varepsilon\} \leq C \varepsilon^{-q} d^p \quad (5.4)$$

for all $\varepsilon > 0$ and $d \in \mathbb{N}$. In particular, there is an $r > 1$ with

$$n(d^{-1}, d) \leq d^r \quad (5.5)$$

for every $d \in \mathbb{N}$. If d is large enough, we can apply Part (ii) of Theorem 4 for $n = d^r$ and the estimate

$$\beta(d, d^r) = \frac{\log \sigma_{2,d}^{-1}}{\log \left(1 + \frac{v \cdot d}{r \log_{1+v} d}\right)} \leq \frac{2 \log \sigma_{2,d}^{-1}}{\log d} \quad (5.6)$$

to obtain

$$d^{-1} \geq a_{n(d^{-1}, d), d} \geq a_{d^r, d} \geq \sigma_{2,d} \cdot d^{-r\beta(d, d^r)} \geq \sigma_{2,d}^{2r+1}. \quad (5.7)$$

Consequently, $\sigma_{2,d}$ decays polynomially in d .

Now let $\sigma_{2,d}$ be of polynomial decay. Then there are constants $p > 0$ and $d_0 \in \mathbb{N}$ such that $\sigma_{2,d}$ is bounded above by d^{-p} for any $d \geq d_0$. On the other hand, there are positive constants C and s such that

$$\sigma_{n,d} \leq \sigma_{n,1} \leq C n^{-s}. \quad (5.8)$$

We apply Part (i) of Theorem 4 and the estimate

$$\alpha(d, 1) = \frac{\log \sigma_{2,d}^{-1}}{\log d + \frac{2}{s} \log \sigma_{2,d}^{-1}} \geq \frac{p}{1 + \frac{2p}{s}} = r > 0 \quad (5.9)$$

to obtain

$$a_{n,d} \leq \left(\frac{\exp(C^{2/s})}{n} \right)^r \quad (5.10)$$

for any $n \in \mathbb{N}$ and $d \geq d_0$. Consequently,

$$n(\varepsilon, d) = \# \{n \in \mathbb{N} \mid a_{n,d} \geq \varepsilon\} \leq \exp(C^{2/s}) \cdot \varepsilon^{-1/r} \quad (5.11)$$

for any $d \geq d_0$ and $\varepsilon > 0$ and $\{T_d\}$ is strongly polynomially tractable. \square

Let us consider the spaces $H_{\text{mix}}^s(\mathbb{I}^d)$ and $H_{\text{mix}}^s(\mathbb{T}^d)$ as defined in Section 4.3. The L_2 -approximation in these spaces is not polynomially tractable. Can we achieve polynomial tractability by increasing the smoothness with the dimension?

Corollary 9. *The problem $\{H_{\text{mix}}^{s_d}(\mathbb{I}^d) \hookrightarrow L_2(\mathbb{I}^d)\}$ is not polynomially tractable for any choice of natural numbers s_d . The problem $\{H_{\text{mix}}^{s_d}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d)\}$ is strongly polynomially tractable, iff it is polynomially tractable, iff $b - a < 2\pi$ and s_d grows at least logarithmically in d or $b - a = 2\pi$ and s_d grows at least polynomially in d .*

With regard to tractability, the L_2 -approximation of mixed order Sobolev functions is hence much harder for nonperiodic than for periodic functions. The negative tractability result for nonperiodic functions can be explained by the difficulty of approximating d -variate polynomials with degree one or less in each variable and H_{mix}^1 -norm less than one. The corresponding set of functions is contained in the unit ball of the nonperiodic space H_{mix}^s for every $s \in \mathbb{N} \cup \{\infty\}$.

Note that Corollary 9 for cubes of unit length is in accordance with [PW10], where Papageorgiou and Woźniakowski prove the corresponding statement for the L_2 -approximation in Sobolev spaces of mixed smoothness (s_1, \dots, s_d) on the unit cube. The smoothness of such functions increases from variable to variable, but the

smoothness with respect to a fixed variable does not increase with the dimension. There, the authors raise the question for a characterization of spaces and their norms for which increasing smoothness yields polynomial tractability. Theorem 5 says that in the setting of uniformly increasing mixed smoothness, polynomial tractability is achieved, if and only if it leads to a polynomial decay of the second singular value of the univariate problem. It would be interesting to verify whether the same holds in the case of variable-wise increasing smoothness and to compute the exponents of strong polynomial tractability.

The reason for the great sensibility of the tractability results for the periodic spaces to the length of the interval can be seen in the difficulty of approximating trigonometric polynomials with frequencies in $\frac{2\pi}{b-a} \{-1, 0, 1\}^d$ that are contained in the unit ball of $H_{\text{mix}}^\infty(\mathbb{T}^d)$. The corresponding set of functions is nontrivial, if and only if $\frac{2\pi}{b-a}$ is smaller than one.

It may yet seem unnatural that the approximation numbers are so sensible to the representation $[\mathbf{a}, \mathbf{b}]$ of the d -torus or the d -cube. This can only happen, since the above and common scalar products

$$\langle f, g \rangle = \sum_{\alpha \in \{0, \dots, s\}^d} \langle D^\alpha f, D^\alpha g \rangle_{L_2} \quad (5.12)$$

do not define a homogeneous family of norms on $H_{\text{mix}}^s([\mathbf{a}, \mathbf{b}])$. To see that, let T_d be the embedding of $H_{\text{mix}}^s([\mathbf{a}, \mathbf{b}])$ into $L_2([\mathbf{a}, \mathbf{b}])$ and let T_d^* be the embedding in the case $[\mathbf{a}, \mathbf{b}] = [0, 1]^d$. The dilation operation $Mf = f(\mathbf{a} + (\mathbf{b} - \mathbf{a}) \cdot)$ defines a linear homeomorphism both from $L_2([\mathbf{a}, \mathbf{b}])$ into $L_2([0, 1]^d)$ and from $H_{\text{mix}}^s([\mathbf{a}, \mathbf{b}])$ into $H_{\text{mix}}^s([0, 1]^d)$ and clearly

$$T_d^* = MT_d M^{-1}. \quad (5.13)$$

The L_2 -spaces satisfy the homogeneity relation

$$\|Mf\|_{L_2([0, 1]^d)} = \lambda^d([\mathbf{a}, \mathbf{b}]) \cdot \|f\|_{L_2([\mathbf{a}, \mathbf{b}])} \quad \text{for } f \in L_2([\mathbf{a}, \mathbf{b}]). \quad (5.14)$$

If the chosen family of norms on $H_{\text{mix}}^s(\mathbb{T}^d)$ is also homogeneous, i.e.

$$\|Mf\|_{H_{\text{mix}}^s([0, 1]^d)} = \lambda^d([\mathbf{a}, \mathbf{b}]) \cdot \|f\|_{H_{\text{mix}}^s([\mathbf{a}, \mathbf{b}])} \quad \text{for } f \in H_{\text{mix}}^s([\mathbf{a}, \mathbf{b}]), \quad (5.15)$$

the approximation numbers of T_d and T_d^* clearly must coincide. The above scalar

products do not yield a homogeneous family of norms. An example of an equivalent and homogeneous family of norms on $H_{\text{mix}}^s([\mathbf{a}, \mathbf{b}])$ is defined by the scalar products

$$\langle f, g \rangle = \sum_{\alpha \in \{0, \dots, s\}^d} (\mathbf{b} - \mathbf{a})^{-2\alpha} \langle D^\alpha f, D^\alpha g \rangle_{L_2}. \quad (5.16)$$

Hence, the approximation numbers and tractability results with respect to this scalar product do not depend on \mathbf{a} and \mathbf{b} at all. They coincide with the approximation numbers with respect to the previous scalar product on $H_{\text{mix}}^s([0, 1]^d)$.

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